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ON AN APPROXIMATE METHOD OF SOLVING INTEGRAL EQUATIONS OF DYNAMIC CONTACT PROBLEMS

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V. A. BABESHKO and V. V. KALINCHUK
(Rostov-on-Don)
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Integral equations which originate in a number of contact problems concerned with the vibrations of stamps on the surface of domains, whose boundaries are at infinity (for example, on the surface of a layer, of a multilayer foundation, of a cylinder, a tube, etc.), are considered. Such problems reduce to integral equations of the first kind with a difference kernel containing oscillating members. The oscillations grow as the vibration frequency increases, and this either makes application of known methods of solving equations of the first kind difficult, or completely excludes such a possibility.

The possibility is studied of using a method of solving these equations and questions of its efficiency are discussed (*). In principle, the method permits construction of exact solutions of some equations approximating the initial equations, and errors in the approximate solutions are given.

1. The problem for an elastic layer of thickness h lying friction-free on a rigid foundation during vibrations of a stamp surface of width $2a$ adhering friction-free to its surface, results in an integral equation of the form

$$\int_{-a}^{a} k(x-t)\,q(t)\,dt = \pi f(x), \quad |x| \leqslant a \tag{1.1}$$

$$k(t) = \int_{\Gamma} K(u)\,e^{iut}du \tag{1.2}$$

$$K(u) = [u^2\sigma_2 \operatorname{cth}\sigma_2 - (u^2 - {}^1/_2\varkappa_2^2)^2\,\sigma_1^2 \operatorname{cth}\sigma_1]^{-1} \tag{1.3}$$

$$\sigma_k = \sqrt{u^2-\varkappa_k^2}, \quad \varkappa_1^2 = \rho\omega^2h^2(2\mu+\lambda)^{-1}, \quad \varkappa_2^2 = \rho\omega^2h^2\mu^{-1}$$

*) A. V. Belokon also expressed the possibility of using this method in one of the seminars of the elasticity theory department of Rostov State University.

Here $q(x)$ are the unknown contact stresses, $f(x)$ is a function describing the motion of the stamp surface ($f(x) \equiv 1$) for vibrations of a plane stamp), λ, μ are Lamé coefficients, ν, ρ, ω are the Poisson's ratio, density of the layer material, and frequency of stamp vibration, respectively.

The problem of the radial vibrations of a rigid belt of width $2a$, set on an elastic cylinder of radius h, also reduces to (1.1) in which the function $K(u)$ is given by the relationship

$$K(u) = -\frac{\varkappa_2^2\sigma_1}{4}\left[\left(u^2 - \frac{\varkappa_2^2}{2}\right)^2 I(\sigma_1) - u^2\sigma_1\sigma_2 I(\sigma_2) + \frac{\varkappa_2^2\sigma_1}{2}\right]^{-1} \quad (1.4)$$

($I_k(x)$ is the k-th order Bessel function of imaginary argument, and $I(u) = I_1^{-1}(u)$ $I_0(u)$).

A characteristic singularity of the function $K(u)$ is the presence of zeros and poles on the real axis, whose number generally grows as the frequency ω increases. In other respects, these functions possess the properties of the functions in corresponding static problems [1], namely, they are real on the real axis, meromorphic in the complex plane and diminish at infinity as $|u^{-1}|$.

The presence of real zeros and poles in the functions $K(u)$ is due to the appearance of elastic waves in corresponding domains, which should have a definite direction in the absence of sources at infinity [2, 3]. (This question has been discussed in [4]). The direction of the elastic waves dictates the selection of the contour Γ in the representation (1.2). As a rule, when the time factor $e^{-i\omega t}$ is chosen, this contour coincides everywhere with the real axis except for segments containing the real poles. The positive poles are here bypassed from below, and the negative from above (*).

The case when the zero or pole of the function $K(u)$ coincides with the origin of reference, as well as the rare case when the residues of $K(u)$ in the positive poles have unlike signs, are not considered here.

On the basis of [4], the integral equation (1.1) is then uniquely solvable for any twice continuously differentiable right side in the space of continuous weighted functions, i.e.

$$\| q(x) \sqrt{a^2 - x^2} \|_C \leqslant N \| f \|_{C^2}$$

In order to construct the approximate solution of (1.1) below, we shall approximate the function $K(u)$ by the approximate function $H_1(u)$ (A is an additional approximation parameter)

$$H_1(u) = H_0(u) \prod_{k=1}^{m} (u^2 - z_k^2)(u^2 - \gamma_k^2)^{-1}, \quad H_0(u) = u^{-1} \operatorname{th} Au \quad (1.5)$$

with the property

$$|K(u) - H_1(u)|\, |K^{-1}(u)| (1 + |u|)^\alpha < \delta, \quad \alpha > 1/2, \quad -\infty \leqslant u \leqslant \infty \quad (1.6)$$

conserved.

Condition (1.6) imposes the requirement that the real zeros and poles of the functions $K(u)$ and $H_1(u)$ agree. This requirement does not extend to the distribution of complex zeros and poles. In this case, for sufficiently small δ , the solution of the integral equations whose kernels are defined by the functions $K(u)$ and $H_1(u)$ will be close

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*) It is shown that exceptions will exist in the note of V. A. Babeshko "New effective method of solving dynamic contact problems" (Abstracts of Reports to the 13th International Congress on Theoretical and Applied Mechanics, Moscow, 1972).

in the sense of [5], i.e.

$$\| (q - q_1) \sqrt{a^2 - x^2} \|_C < M\delta \| q \sqrt{a^2 - x^2} \|_C \tag{1.7}$$

The constant M depends only on the function $K(u)$.

We therefore arrive at the need to solve (1.1) with the kernel (1.2) in which $H_1(u)$ plays the part of the function $K(u)$.

As an example, let us present the function $H_1(u)$ in (1.5) which approximates the function $K(u)$ in (1.4) with error no exceeding 10% for $\nu = 0.3, \varkappa_2 = 4.45, A = 0.1$. In this case, there are three poles on the real axis: $\gamma_1 = 0.7523$, $\gamma_2 = 1.3575$, $\gamma_3 = 4.4956$, four complex poles: $\gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = 7.774i$, and also two real zeros: $z_1 = 2.2629$, $z_2 = 2.3786$, and the remaining zeros are complex: $z_3 = 2.9270\,i$, $z_4 = -1.2870 + 7.6493\,i$, $z_5 = 1.2870 + 7.6493i$, $z_6 = 7.7684i$, $z_7 = 11i$.

2. Let us assume

$$h_n(u) = \int_{-\infty}^{\infty} H_n(t)\, e^{iut} dt, \quad n = 0, 1 \tag{2.1}$$

Equation (1.1) with the kernel $h_0(u)$ is solved exactly [6, 7].

Let us reduce the solution of (1.1) with the kernel $h_1(u)$ to the solution of the same equation with the kernel $h_0(u)$.

To this end, let us use a representation of the form [8]

$$\int_{-a}^{a} h_1(x - t)\, q_1(t)\, dt = \prod_{k=1}^{m} \left(-\frac{d^2}{dx^2} - z_k^2\right) R(x) = \pi f(x) \tag{2.2}$$

$$R(x) = \int_{-a}^{a} \int_{-\infty}^{\infty} H_0(u) \prod_{k=1}^{m} (u^2 - \gamma_k^2)^{-1} e^{iu(x-t)} du\, q(t)\, dt$$

The last equation in (2.2) is an inhomogeneous differential equation with constant coefficients of order $2m$ in the function $R(x)$. Having determined its general solution for $f(x) = \exp i\eta x$, and then having applied a differential operator of the form

$$M\left(\frac{d}{dx}\right) = \prod_{k=1}^{m} \left(-\frac{d^2}{dx^2} - \gamma_k^2\right)$$

we arrive at an equation to determine the unknown $q(x)$ representable as

$$\int_{-a}^{a} h_0(x - t)\, q(t)\, dt = \pi S(\eta)\, e^{i\eta x} + \sum_{k=1}^{m} (x_k^- e^{-iz_k x} + x_k^+ e^{iz_k x}) \tag{2.3}$$

$$S(\eta) = \prod_{k=1}^{m} (\eta^2 - z_k^2)(\eta^2 - \gamma_k^2)^{-1}$$

The constants $x_k^{\pm}$ in the representation (2.3) are to be determined from the condition that the function $q(x)$ found must satisfy the initial equation (1.1). Equation (2.3) with an arbitrary right side $f(x)$ is solved in closed form [7]. For the purposes herein, it is sufficient to solve this equation for $f(x) = \exp i\eta\, x$. Denoting the solution of this equation by $q_0(x)$, we obtain

$$q_0(x,\ \eta) = (2A)^{-1}e^{-ba}P(a-x)P(a+x)[M - i\eta b^{-1}J(x)] \tag{2.4}$$

$$J(x) = be^{ba}\int_{-a}^{a} P^{-1}(-a+t)P^{-1}(a+t)P^{-2}(t-x)e^{i\eta t}dt$$

$$M = e^{ba}K^{-1}(e^{-2ba})\left[\frac{\pi}{2}e^{-i\eta a} + i\eta be^{ba}\int_{-\infty}^{a} P(a-\xi)P^{-1}(-a-\xi)\times\right.$$

$$\left. e^{b\xi}\int_{-a}^{a} P^{-1}(a-t)P^{-1}(a+t)P^{-2}(t-\xi)e^{i\eta t}dt\,d\xi\right]$$

$$P(t) = (1-e^{-2bt})^{-1/2},\qquad b = (2A)^{-1}\pi$$

Here $K(x)$ is the complete elliptic integral of the first kind.

Therefore, the solution of (2.2) with the right side mentioned is, because of (2.3), (2.4)

$$q_1(x) = \pi S(\eta)q_0(x,\eta) + \sum_{k=1}^{m}[c_kq_0(x,-iz_k) + d_kq_0(x,iz_k)] \tag{2.5}$$

On the other hand, the function $q_1(x)$ can be represented as

$$q_1(x) = Y_0e^{i\eta x} + \sum_{k=-m}^{\infty}(Y_k^+e^{i\eta_k(a+x)} + Y_k^-e^{i\eta_k(a-x)}) \tag{2.6}$$

The constants $Y_k\pm$ are here determined from the infinite system in which it is necessary to take the upper and lower signs in sequence

$$\frac{Y_0e^{\mp ia\eta}}{\xi_k\mp\eta} + \sum_{l=-m}^{\infty}\left(\frac{Y_l^\pm}{\xi_k-\eta_l} + \frac{Y_l^\pm}{\xi_k+\eta_l}\right) = 0 \tag{2.7}$$

$$k = -m,\ -m+1,\ \ldots,\ -1,\ 1,\ \ldots$$

$$\xi_k = \gamma_{-k},\quad k=-1,\ -2,\ \ldots -m,\quad \xi_k = i(2k-1)b,\quad k=1,\ 2,\ \ldots$$

The function $q_0(x,\eta)$ can also be represented as (2.6), namely:

$$q_0(x,\eta) = \frac{i\eta}{2}\operatorname{ctg}(i\eta A)e^{i\eta x} + \sum_{k=0}^{m}[c_k^+(\eta)e^{i\eta_k(a+x)} + c_k^-(\eta)e^{i\eta_k(a-x)}] \tag{2.8}$$

and hence the coefficients $c_k\pm(\eta)$ are determined exactly as a result of expanding the left side of (2.7) in an analogous series. These coefficients are

$$2\pi C_k^\pm(\eta) = i\eta(e^{i\eta a}\gamma_k^\pm - e^{-i\eta a}\gamma_{1k}^\pm) + (i\eta e^{-i\eta a}\beta_{10} + 2bMe^{-ba})\gamma_{2k}^\pm$$

$$\gamma_k^+ = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\beta_{2p}\delta_n\delta_{n+p+k}B(p+n),\qquad \gamma_k^- = \sum_{n=k}^{\infty}\sum_{p=0}^{n}\beta_{2p}\delta_{n-p}\delta_{n-k}B(n-k)$$

$$\gamma_{1k}^+ = \sum_{n=k}^{\infty}\sum_{p=0}^{n}\beta_{1p}\delta_{n-p}\delta_{n-k}B(n-k),\qquad \gamma_{1k}^- = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\beta_{1p}\delta_n\delta_{n+p+k}B(p+n)$$

$$\gamma_{2k}^\pm = \sum_{n=0}^{\infty}\delta_n\delta_{n+k}B(n),\qquad \beta_{1p} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\alpha_n\alpha_kB(n)[n-k+p+i\eta A\pi^{-1}]^{-1}$$

$$\beta_{2p} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \alpha_n \alpha_k B(n) [n - k + p - i\eta A\pi]^{-1}$$

$$\alpha_0 = \delta_0 = 1, \quad -\alpha_1 = \delta_1 = \frac{1}{2}$$

$$\delta_n = -(2n-1)\alpha_n = \frac{(2n-1)!!}{2n!!}, \quad n = 2, 3, \ldots$$

$$B(n) = e^{-4bna}, \qquad b = (2A)^{-1}\pi$$

They are the solutions of the infinite system (2.7) from which the first m rows and columns must be discarded, i.e. they satisfy the system

$$\frac{\theta e^{\mp i\eta a}}{2(\xi_r \mp \eta)} + \sum_{k=1}^{\infty} \left[\frac{c_k^{\mp}(\eta)}{\xi_r - \eta_k} + \frac{c_k^{\pm}(\eta)}{\xi_r + \eta_k} e^{2ai\eta_k} \right] = 0, \quad r = 1, 2, \ldots \qquad (2.9)$$

where $\theta = i\eta \operatorname{ctg}(i\eta A)$.

Taking this circumstance into account, we insert (2.8) into (2.5) and easily establish the connection between the coefficients $Y_l^{\pm}$ and the unknown constants $x_p^{\pm}$ and $c_k^{\pm}(\eta)$. This connection is given by the relationship

$$Y_{-p}^{\pm} = (2\pi)^{-1} iz_p e^{-z_p a} \operatorname{ctg}(iz_p A) x_p^{\pm}, \quad p = 1, 2, \ldots m \qquad (2.10)$$

$$Y_k^{\pm} = \pi^{-1} \sum_{p=0}^{m} [x_p^{\pm} c_k^{\pm}(-iz_p) + x_p^{\mp} c_k^{\mp}(iz_p)], \quad k = 1, 2 \ldots$$

Table 1

$\varkappa_2$	γ_k	z_k
1.9	1.2324	1.0156
3.2	2.6838	1.7104
5.8	2.1840 3.3969 6.1333	3.1002 3.1002 4.3079
7.1	1.0349 3.4447 4.7459 7.6136	 1.0915 3.7951 5.9773
8.4	3.4747 4.5143 6.5099 9.0573	2.3405 4.4900 5.0634 7.4766

Now, let us insert the relationship (2.10) into (2.7) and let us take into account that the constants $c_k^{\pm}(\eta)$ become identical to the corresponding systems (2.9) for $\eta = \pm iz_p$, $p = 1, 2, \ldots m$.

As a result of manipulation, we arrive at the definition of the unknowns $x_p^{\pm}$ of a finite system of linear algebraic equations of the form

$$\sum_{p=1}^{m} [A_r^{\pm}(-iz_p) x_p^{+} + A_r^{\pm}(iz_p) x_p^{-}] =$$

$$-\pi S(\eta) A_r^{\pm}(\eta), \quad r = 1, 2, \ldots, m$$

where $(A_r^{\pm}(\eta)$ are the left sides of (2.9).

3. Presented in Table 1 as an example is the distribution of real zeros and poles of the function $K(u)$ given by (1.4) for $\nu = 0.3$ as a function of the frequency (of values of the parameter $\varkappa_2$). The distribution of real zeros and poles of the function $K(u)$ given by (1.3) and closely related to the Rayleigh function is well known in the literature [9] and is not presented here.

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EXACT SOLUTION OF THE EXTERIOR FUNDAMENTAL MIXED PROBLEM

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V. I. FABRIKANT
(Ul'ianovsk)
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An exact solution of the fundamental exterior mixed problem with a circular line of separation of the boundary conditions for a transversely isotropic half-space is proposed. The interior fundamental mixed problem for an isotropic half-space has been examined in [1, 2].

1. Let us consider a transversely isotropic half-space $z \geqslant 0$ whose planes of isotropy are parallel to the boundary. We understand the problem with the following conditions on the boundary $z = 0$:

$$\sigma_z = \sigma(\rho, \varphi), \qquad \tau_{zx} = \tau_{zx}(\rho, \varphi)$$
$$\tau_{yz} = \tau_{yz}(\rho, \varphi) \qquad (\rho \leqslant a)$$
$$w = w(\rho, \varphi), \qquad u_x = u_x(\rho, \varphi)$$
$$u_y = u_y(\rho, \varphi) \qquad (\rho > a)$$

to be the exterior fundamental mixed problem.

We introduce the complex tangential displacements $u = u_x + iu_y$ and shear stresses $\tau = \tau_{zx} + i\tau_{yz}$, $\overline{\tau} = \tau_{zx} - i\tau_{yz}$. If the requirement of decomposability in Fourier series in the angular coordinate is imposed on the given and desired functions, then